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MASTER EQUATIONS FOR MASTER AMPLITUDES*

M. Caffo[a,b], H. Czyż[c], S. Laporta[b] and E. Remiddi[b,a]

[a] INFN, Sezione di Bologna, I-40126 Bologna, Italy
[b] Dipartimento di Fisica, Università di Bologna, I-40126 Bologna, Italy
[c] Institute of Physics, University of Silesia, PL-40007 Katowice, Poland
e-mail: caffo@bo.infn.it, czyz@usctoux1.cto.us.edu.pl,
laporta@bo.infn.it, remiddi@bo.infn.it.

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The general lines of the derivation and the main properties of the master equations for the master amplitudes associated to a given Feynman graph are recalled. Some results for the 2-loop self-mass graph with 4 propagators are presented.

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1. Introduction

The integration by part identities [1] are by now a standard tool for obtaining relations between the many integrals associated to any Feynman graph or, equivalently, for working out recurrence relations for expressing the generic integral in terms of the "master integrals" or "master amplitudes" of the considered graph. A good example of the use of the integration by part identities is given in [2], where the recurrence relations for all the 2-loop self-mass amplitudes are established in the case of arbitrary masses.

It has been shown in [3] that by that same technique one can obtain a set of linear first order differential equations for the master integrals themselves. The coefficients of the equations are a ratio of polynomials with integer coefficients in all the variables and the equations are further non-homogeneous, with the non-homogeneous terms given by the master integrals of the simpler graphs obtained from the considered graph by removing one or more internal propagators.

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Restricting ourselves for simplicity to the self-mass case, for any Feynman graph the related integrals can in general be written in the form

$$A(\alpha, p^2) = \int d^n k \; B(\alpha, p, k) \; . \tag{1}$$

In more detail, $d^n k = d^n k_1 ... d^n k_l$ stands for the n-continuous integration on an arbitrary number l of loops and k stands for the set corresponding loop momenta, so that there are all together $s = (l+1)(l+2)/2$ different scalar products, including p^2. $B(\alpha, p, k)$ is the product of any power of the scalar products in the numerators divided by any power of the propagators occurring in the graph (all masses will always be taken as different, unless otherwise stated). As the propagators are also simple combinations of the scalar products, simplifications might occur between numerator and denominator and as a consequence one expects quite in general $(s-1)$ different factors altogether in the numerator and denominator, independently of the actual number of propagators present in the graph. Graphs with less propagators have more factors in the numerator and *viceversa*. Therefore, the symbol α in Eq. (1) stands in fact for a set of $(s-1)$ indices – the (integer) powers of the $(s-1)$ factors. The integration-by-parts corresponding to the amplitudes of Eq. (1) are

$$\int d^n k \frac{\partial}{\partial k_{i,\mu}} \Big[v_\mu B(\alpha, p, k) \Big] = 0 \; , \qquad i = 1, .., l \tag{2}$$

where v stands for any of the $(l+1)$ vectors k and p. There are therefore $l(l+1)$ identities for each set of indices α. The identity is easily established. For small n the integral of the divergence vanishes. When the derivatives are explicitly carried out, one obtains the sum of a number of terms, all equal to a simple coefficient (an integer number or, occasionally, n), times an integrand of the form $B(\beta, k, p)$, with the set of indices β differing at most by a unity in two places from the set α. That set of the identities is infinite; even if they are not all independent, they can be used for obtaining the recurrence relations by which one can express each integral in terms of a few already mentioned "master amplitudes" through a relation of the form

$$A(\alpha, p^2) = \sum_m C(\alpha, m) A(m, p^2) + \sum_j C(\alpha, j) A(j, p^2) \; , \tag{3}$$

where the set of indices m takes the very few values corresponding to the master amplitudes. j refers to simpler master integrals in which one or more denominators are missing and the coefficients $C(\alpha, m), C(\alpha, j)$ are ratios of polynomials in n, the masses and p^2.

Let us consider now one of the master amplitudes themselves, say the master amplitude identified by the set of indices m. According to Eq. (1) we can write

$$A(m,p^2) = \int d^nk \; B(m,p,k) \; . \tag{4}$$

By acting with $p_\mu(\partial/\partial p_\mu)$ on both sides we get

$$p^2 \frac{\partial}{\partial p^2} A(m,p^2) = \frac{1}{2} \int d^nk \; p_\mu \frac{\partial}{\partial p_\mu} \; B(m,p,k) \; . \tag{5}$$

According to the discussion following Eq. (2) the is a combination of integrands. As Eq. (3) applies to each of the corresponding integrals one obtains the relations

$$p^2 \frac{\partial}{\partial p^2} A(m,p^2) = \sum_{m'} C(m,m') A(m',p^2) + \sum_j C(m,j) A(j,p^2) \; , \tag{6}$$

which are the required master equations. As in Eq. (3), j refers to simpler master integrals (in which one or more denominator are missing; they constitute the non-homogeneous part of the master equations), to be considered as known when studying the $A(m,p^2)$. It is obvious from the derivation that the master equations can be established regardless of the number of loops. It is equally clear that for graphs depending on several external momenta (such as vertex or 4-body scattering graphs) one has simply to replace the single operator $p_\mu(\partial/\partial p_\mu)$ of Eq. (5) by the set of operators $p^i_\mu(\partial/\partial p^j_\mu)$, where i,j run on all the external momenta, and with some more algebra one can obtain master equations in any desired Mandelstam variable. The master equations are a powerful tool for the study and the evaluation of the master amplitudes; among other things:

- They provide information on the values of the master amplitudes at special kinematical points (such as $p^2 = 0$ in Eq. (6); the l.h.s. vanishes, as $p^2 = 0$ is a regular point, so that the r.h.s. is a relation among master amplitudes at $p^2 = 0$, usually sufficient to fix their values at that point).

- The master equations are valid identically in n, so that they can be expanded in $(n-4)$ and solved recursively for the various terms of the expansion in $(n-4)$ starting from the most singular one (for 2-loop amplitudes one expects at most a double pole in $(n-4)$).

- When the initial value at $p^2 = 0$ has been obtained, the equations can be integrated by means of fast and precise numerical methods (for instance by a Runge-Kutta routine), providing a convenient approach to their numerical evaluation. Note that the numerical approach can be used both for arbitrary n or for $n = 4$, once the expansion has been properly carried out.

- The equations can be used to work out virtually any kind of expansion, in particular the large p^2 expansion, as will be shown in some detail.
- In particular simple cases (for instance when most of the masses vanish and only one or two scales are left) the analytic quadrature of the equations can lead to the analytic evaluation of the master amplitudes.

2. The 2-loop 4-propagator graph

The use of the master equations for studying the 1-loop self-mass and the 2-loop sunrise self-mass graph has been already discussed, [3, 4]. We will describe here its application to the 2-loop 4-propagator self-mass graph shown in Fig.1.

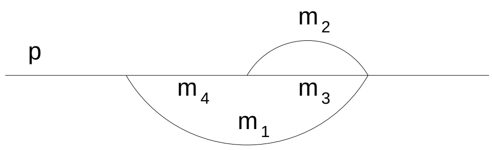

Fig. 1. $G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2)$, the 2-loop 4-propagator self-mass graph.

The corresponding amplitude is defined as

$$G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2) = \frac{1}{C^2(n)} \int \frac{d^n k_1}{(2\pi)^{n-2}} \frac{d^n k_2}{(2\pi)^{n-2}}$$
$$\times \frac{1}{(k_1^2 + m_1^2)\ [(p-k_1)^2 + m_4^2]\ (k_2^2 + m_2^2)\ [(p-k_1-k_2)^2 + m_3^2]}. \quad (7)$$

The conventional factor $C(n)$ is given in [4]. It is sufficient to know that at $n = 4$ its value is 1. If all momenta are Euclidean the $i\epsilon$ in the propagators is not needed. Skipping details, the master equation reads

$$\begin{aligned}
&R^2(-p^2, m_1^2, m_4^2)\, p^2 \frac{\partial}{\partial p^2} G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2) \\
&= \frac{n-4}{2} R^2(-p^2, m_1^2, m_4^2) G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2) \\
&\quad + (n-3) \left[(m_1^2 + m_4^2)p^2 + (m_1^2 - m_4^2)^2\right] G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2) \\
&\quad + (3p^2 - m_1^2 + m_4^2) m_1^2 F_1(n, m_1^2, m_2^2, m_3^2, p^2) \\
&\quad + (p^2 - m_1^2 + m_4^2) \Bigg[\frac{3n-8}{2} F_0(n, m_1^2, m_2^2, m_3^2, p^2)
\end{aligned}$$

$$+ m_2^2 F_2(n, m_1^2, m_2^2, m_3^2, p^2) + m_3^2 F_3(n, m_1^2, m_2^2, m_3^2, p^2)$$
$$\left. - \frac{1}{2}(n-2) V(n, m_2^2, m_3^2, m_4^2) \right], \tag{8}$$

where $F_k(n, m_1^2, m_2^2, m_3^2, p^2), k = 0, .., 3$ are the 2-loop self-mass sunrise master amplitudes [2, 4],

$$F_0(n, m_1^2, m_2^2, m_3^2, p^2) = \frac{1}{C^2(n)} \int \frac{d^n k_1}{(2\pi)^{n-2}} \frac{d^n k_2}{(2\pi)^{n-2}}$$
$$\times \frac{1}{(k_1^2 + m_1^2)\ (k_2^2 + m_2^2)\ [(p - k_1 - k_2)^2 + m_3^2]} \tag{9}$$

and

$$F_i(n, m_1^2, m_2^2, m_3^2, p^2) = -\frac{\partial}{\partial m_i^2} F_0(n, m_1^2, m_2^2, m_3^2, p^2), \quad i = 1, 2, 3\ , \tag{10}$$

while $V(n, m_1^2, m_2^2, m_3^2)$ corresponds to the 2-loop vacuum amplitude,

$$V(n, m_1^2, m_2^2, m_3^2) = \frac{1}{C^2(n)} \int \frac{d^n k_1}{(2\pi)^{n-2}} \frac{d^n k_2}{(2\pi)^{n-2}}$$
$$\times \frac{1}{(k_1^2 + m_1^2)\ (k_2^2 + m_2^2)\ [(k_1 + k_2)^2 + m_3^2]} \tag{11}$$

and, as usual,

$$R^2(-p^2, m_1^2, m_2^2) = p^4 + m_1^4 + m_2^4 + 2m_1^2 p^2 + 2m_2^2 p^2 - 2m_1^2 m_2^2\ .$$

The value at $p^2 = 0$ is almost trivially found to be

$$G(n, m_1^2, m_2^2, m_3^2, m_4^2, 0) = \frac{V(n, m_2^2, m_3^2, m_4^2) - V(n, m_1^2, m_2^2, m_3^2)}{m_1^2 - m_4^2}. \tag{12}$$

The expansion in $(n-4)$ reads

$$G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2) = \sum_{k=-2}^{\infty} (n-4)^k G^{(k)}(m_1^2, m_2^2, m_3^2, m_4^2, p^2). \tag{13}$$

By expanding in the same way all the other amplitudes occurring in the master equation Eq. (8) and using the results of [4], the first values are found to be

$$\begin{aligned} G^{(-2)}(m_1^2, m_2^2, m_3^2, m_4^2, p^2) &= +\frac{1}{8}\,, \\ G^{(-1)}(m_1^2, m_2^2, m_3^2, m_4^2, p^2) &= -\frac{1}{16} - \frac{1}{2} S^{(0)}(m_1^2, m_4^2, p^2)\,, \end{aligned} \tag{14}$$

where $S^{(0)}(m_1^2, m_4^2, p^2)$ is the finite part at $n = 4$ of the 1-loop self mass; more exactly, defining

$$S(n, m_1^2, m_2^2, p^2) = \frac{1}{C(n)} \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{(k^2 + m_1^2)\, [(p-k)^2 + m_2^2]} \,,$$

and expanding in $(n-4)$, one finds [3]

$$S(n, m_1^2, m_2^2, p^2) = -\frac{1}{2}\frac{1}{(n-4)} + S^{(0)}(m_1^2, m_2^2, p^2) + \mathcal{O}(n-4) \tag{15}$$

with

$$\begin{aligned} S^{(0)}(m_1^2, m_2^2, p^2) &= \frac{1}{2} - \frac{1}{4}\ln(m_1 m_2) \\ &+ \frac{1}{4p^2}\left[R(-p^2, m_1^2, m_2^2)\ln(u(p^2, m_1^2, m_2^2)) + (m_1^2 - m_2^2)\ln\frac{m_1}{m_2}\right], \end{aligned} \tag{16}$$

where

$$\begin{aligned} R(-p^2, m_1^2, m_2^2) &= \sqrt{[p^2 + (m_1+m_2)^2][p^2 + (m_1-m_2)^2]}\,, \\ u(p^2, m_1^2, m_2^2) &= \frac{\sqrt{p^2 + (m_1+m_2)^2} - \sqrt{p^2 + (m_1-m_2)^2}}{\sqrt{p^2 + (m_1+m_2)^2} + \sqrt{p^2 + (m_1-m_2)^2}}\,. \end{aligned}$$

3. The large p^2 expansion

Quite in general, if $\omega = (n-4)/2$, the large p^2 expansion is

$$\begin{aligned} G(n, m_1^2, m_2^2, m_3^2, m_4^2, p^2) &= (p^2)^{2\omega} \sum_{k=0}^{\infty} G_k^{(\infty,2)}(n, m_1^2, m_2^2, m_3^2, m_4^2)\frac{1}{(p^2)^k} \\ &+ (p^2)^{\omega} \sum_{k=0}^{\infty} G_k^{(\infty,1)}(n, m_1^2, m_2^2, m_3^2, m_4^2)\frac{1}{(p^2)^k} \\ &+ \frac{1}{p^2} \sum_{k=0}^{\infty} G_k^{(\infty,0)}(n, m_1^2, m_2^2, m_3^2, m_4^2)\frac{1}{(p^2)^k}\,. \end{aligned} \tag{17}$$

Indeed any l-loop amplitude at large p^2 develops l terms with "fractional powers" in p^2, with exponents $\omega, 2\omega, \cdots l\omega$, besides the "regular" term containing integer powers only. As any 2-loop amplitude has "fractional dimension" equal to 2 (in square mass units), the coefficients $G_k^{(\infty,2)}, G_k^{(\infty,1)}$ and

$G_k^{(\infty,0)}$ must have "fractional dimension" equal to 0, 1 and 2 respectively, on dimensional grounds. Similar expansions are valid for the sunrise amplitudes appearing in the r.h.s. of Eq. (8) such as

$$\begin{aligned} F_0(n, m_1^2, m_2^2, m_3^2, p^2) &= p^2(p^2)^{2\omega} \sum_{k=0}^{\infty} F_{0,k}^{(\infty,2)}(n, m_1^2, m_2^2, m_3^2) \frac{1}{(p^2)^k} \\ &+ (p^2)^{\omega} \sum_{k=0}^{\infty} F_{0,k}^{(\infty,1)}(n, m_1^2, m_2^2, m_3^2) \frac{1}{(p^2)^k} \\ &+ \frac{1}{p^2} \sum_{k=0}^{\infty} F_{0,k}^{(\infty,0)}(n, m_1^2, m_2^2, m_3^2) \frac{1}{(p^2)^k} \end{aligned} \tag{18}$$

as well as for the 1-loop self-mass amplitude whose "fractional power" is just ω

$$\begin{aligned} S(n, m_1^2, m_2^2, p^2) &= (p^2)^{\omega} \sum_{k=0}^{\infty} S_k^{(\infty,1)}(n, m_1^2, m_2^2) \frac{1}{(p^2)^k} \\ &+ \frac{1}{p^2} \sum_{k=0}^{\infty} S_k^{(\infty,0)}(n, m_1^2, m_2^2) \frac{1}{(p^2)^k} \ . \end{aligned} \tag{19}$$

When the above expansions are inserted into the master equations and the recurrence relations, to be regarded, in this context, as differential equations in the masses, one obtains a number of equations providing important relations between the coefficients of the expansions. As a result, one finds [3]

$$S_0^{(\infty,1)}(n, m_1^2, m_2^2) = S^{(\infty)}(n) \ , \tag{20}$$

where $S^{(\infty)}(n)$ is a dimensionless function of n only. An explicit calculation (most easily performed by putting $m_2 = 0$) yields for its expansion in $n-4$

$$S^{(\infty)}(n) = -\frac{1}{2}\,\frac{1}{n-4} + \frac{1}{2} + \left(\frac{1}{8}\zeta(2) - \frac{1}{2}\right)(n-4) + \mathcal{O}\left((n-4)^2\right) . \tag{21}$$

All the other terms $S_k^{(\infty,1)}(n, m_1^2, m_2^2), k = 1, 2, ..$ can then obtained explicitly and are found to be proportional to $S^{(\infty)}(n)$. One further finds

$$S_0^{(\infty,0)}(n, m_1^2, m_2^2) = \frac{m_1^{n-2} + m_2^{n-2}}{(n-2)(n-4)} \tag{22}$$

and similar expressions for all the other $S_k^{(\infty,0)}$. Note here that in the massless limit one has the exact relation

$$S(n, 0, 0, p^2) = (p^2)^{\omega} S^{(\infty)}(n) \ . \tag{23}$$

Likewise, one obtains[1].

$$F_{0,0}^{(\infty,2)}(n,m_1^2,m_2^2,m_3^2) = F^{(\infty,2)}(n)\,, \tag{24}$$

$$F_{0,0}^{(\infty,1)}(n,m_1^2,m_2^2,m_3^2) = F^{(\infty,1)}(n)\left(m_1^{n-2}+m_2^{n-2}+m_3^{n-2}\right)\,, \tag{25}$$

$$F_{0,0}^{(\infty,0)}(n,m_1^2,m_2^2,m_3^2) = \frac{(m_1m_2)^{n-2}+(m_1m_3)^{n-2}+(m_2m_3)^{n-2}}{(n-2)^2(n-4)^2}\,. \tag{26}$$

By expanding as usual around $n=4$

$$F^{(\infty,i)}(n) = \sum_{j=-2}^{\infty}(n-4)^j F^{(i,j)}\,, \tag{27}$$

one finds

$$\begin{array}{llll} F^{(2,-2)} & = 0 & F^{(1,-2)} & = -\frac{1}{4} \\ F^{(2,-1)} & = \frac{1}{32} & F^{(1,-1)} & = \frac{3}{8} \\ F^{(2,0)} & = -\frac{13}{128}F^{(1,0)} & & = -4F^{(2,1)} \quad + \frac{59}{128}\,. \end{array}$$

When the large p^2 expansions, Eqs (17)–(19), are substituted into Eq. (8), one can express the $G_k^{(\infty,i)}(n,m_1^2,m_2^2,m_3^2,m_4^2)$ in terms of the other coefficients, already known.

One finds

$$G_0^{(\infty,0)}(n,m_1^2,m_2^2,m_3^2,m_4^2) = V(n,m_2^2,m_3^2,m_4^2) \tag{28}$$

and

$$G_0^{(\infty,2)}(n,m_1^2,m_2^2,m_3^2,m_4^2) = \frac{3n-8}{n-4}F^{(\infty,2)}(n)\,, \tag{29}$$

while $G_0^{(\infty,1)}(n,m_1^2,m_2^2,m_3^2,m_4^2)$ depends on the combination

$$(n-2)F^{(\infty,1)}(n) - S^{(\infty)}(n)/(n-4)\,.$$

When the combination is expanded around $n=4$ as in Eq. (27) and the explicit values of Eq. (3) are used, the first 3 terms of the expansion, i.e., the double pole, the simple pole and the term constant in $(n-4)$ are all found to vanish, suggesting the existence of the exact relation

$$F^{(\infty,1)}(n) = \frac{1}{(n-2)(n-4)}S^{(\infty)}(n)\,. \tag{30}$$

[1] The results are reported here with minor changes of notation if compared to [4]

The above result seems to be confirmed by a preliminary investigation of the large p^2 behaviour of the 5-propagator 2-loop self-mass graph. When Eq. (30) is taken as valid one finds

$$G_0^{(\infty,1)}(n, m_1^2, m_2^2, m_3^2, m_4^2) = 0\,, \tag{31}$$

and

$$\begin{aligned} &G_1^{(\infty,1)}(n, m_1^2, m_2^2, m_3^2, m_4^2) \\ &= \frac{S^{(\infty)}(n)}{(n-2)(n-4)}\Big[(m_1^2)^\omega - (n-3)\left((m_2^2)^\omega + (m_3^2)^\omega\right)\Big]\,. \end{aligned} \tag{32}$$

As in previous work, the algebra needed in all the steps of the work has been processed by means of the computer program `FORM` [5] by J. Vermaseren. One of the authors (E.R.) is glad to acknowledge an interesting discussion with K.G. Chetyrkin on the universality of the coefficients of the large p^2 expansions.

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